PROBLEM SET 5. DUE TUESDAY, 12 SEPTEMBER

Supplementary reading. Simmons, Sections 5.1–5.3.

(1) (2pts) Approximate the following numbers, using the tangent line approximation.
   (a) \( \sqrt[3]{28} \)
      \[ (x + \delta x)^\frac{1}{3} \approx x^\frac{1}{3} + (\delta x)(\frac{1}{3}x^{-\frac{2}{3}}) \]
      Substituting \( x = 27 \) and \( \delta x = 1 \), we get:
      \[ 28^{\frac{1}{3}} \approx 27^{\frac{1}{3}} + (1)(\frac{1}{3}27^{-\frac{2}{3}}) \approx 3.037 \]
   (b) \( \sqrt[2]{102} \)
      \[ (x + \delta x)^\frac{1}{2} \approx x^\frac{1}{2} + (\delta x)(\frac{1}{2}x^{-\frac{1}{2}}) \]
      Substituting \( x = 100 \) and \( \delta x = 2 \), we get:
      \[ 102^{\frac{1}{2}} \approx 100^{\frac{1}{2}} + (2)(\frac{1}{2}100^{-\frac{1}{2}}) = 10.1 \]

(2) (2pts) Find the Taylor series (at \( x = 0 \)) for \( f(x) = \frac{1}{1-x} \).

   The \( n \)'th derivative of \( f(x) = \frac{1}{1-x} = (1-x)^{-1} \) is \(-1^n n!(1-x)^{-n}\). The \( n! \)'s in the derivatives cancel the \( n! \)'s in the denominators of the terms of the Taylor series, and at \( x = 0, (1-x)^{-n} = 1 \) for all \( n \). Therefore, our Taylor series is given by:
   \[
   f(x) \approx 1 - x + x^2 - x^3 + x^4 + \ldots
   = \sum_{n=0}^{\infty} (-1)^n x^n
   \]

(3) (4pts) A sphere of radius \( r \) has volume
   \[ V(r) = \frac{4}{3}\pi r^3, \]
   and surface area
   \[ A(r) = 4\pi r^2. \]

   Approximate the volume and surface area of a sphere of radius 7.02 cm.
   You can check your answer by using a calculator to compute the volume
   and surface area exactly.

   Using linear approximations, we have:
   \[ V(r + \delta r) \approx \frac{4}{3}\pi r^3 + (\delta r)(4\pi r^2), \quad A(r + \delta r) \approx 4\pi r^2 + (\delta r)(8\pi r) \]

   Substituting \( r = 7 \) and \( \delta r = .02 \):
\begin{align*}
V(7.02) &\approx \frac{4}{3}(\pi)(7^3) + .02(4\pi)(7^2) \approx 1449 \text{ cm}^3 \\
A(7.02) &\approx 4\pi(7^2) + .02(8\pi)(7) \approx 619.3 \text{ cm}^2
\end{align*}

In contrast, the volume and area computed directly via calculator are approximately 1449 \text{ cm}^3 and 619.2 \text{ cm}^2 respectively. These approximations are quite good.

(4) (4pts) Compute the following integrals.
(a) \( \int x^3 \, dx \)
\[
\int x^3 \, dx = \frac{1}{4}x^4 + C
\]
(b) \( \int \sin(x) \, dx \)
\[
\int \sin(x) \, dx = \cos(x) + C
\]
(c) \( \int e^x \, dx \)
\[
\int e(x) \, dx = e(x) + C
\]
(d) \( \int \sqrt{x} \, dx \)
\[
\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}}
\]

(5) (4pts) Compute the following integrals by substitution, using the substitution given.
(a) \( \int \sqrt{5 + 7x} \, dx, \ u = 5 + 7x \)
\[
u = 5 + 7x \rightarrow du = 7dx \rightarrow dx = \frac{1}{7}du
\]
\[
\int \sqrt{5 + 7x} \, dx = \frac{1}{7} \int \sqrt{u} \, du
= \frac{1}{7} \left( \frac{2}{3}u^{\frac{3}{2}} \right) + C
= \frac{3}{14}(5 + 7x)^{\frac{3}{2}} + C
\]
(b) \( \int \frac{2x}{\sqrt{3+x^2}} \, dx, \ u = \sqrt{3 + x^2} \)
\[
u = \sqrt{3 + x^2} \rightarrow du = \frac{1}{2}(3 + x^2)^{-\frac{1}{2}}(2x)dx
\]
\[
\int \frac{2x}{\sqrt{3+x^2}} \, dx = \int du
= u + C
= \sqrt{3 + x^2} + C
\]
(c) \[ \int 2xe^{x^2} \, dx, \quad u = x^2 \]
\[ u = x^2 \rightarrow du = 2x \, dx \]
\[ \int 2xe^{x^2} \, dx = \int e^u \, du \]
\[ = e^u + C \]
\[ = e^{x^2} + C \]

(d) \[ \int \frac{dx}{(x-4)^5}, \quad u = x - 4 \]
\[ u = x - 4 \rightarrow du = dx \]
\[ \int \frac{dx}{(x-4)^5} = \int \frac{du}{u^5} \]
\[ = u^{-5} \, du \]
\[ = \frac{1}{4} u^{-4} + C \]
\[ = -\frac{1}{4} (x-4)^{-4} + C \]

(6) (4pts) Integrals satisfy
\[ \int (f(x) + g(x)) \, dx = (\int f(x) \, dx) + (\int g(x) \, dx), \]
just like derivatives do. Again like derivatives, they do not satisfy a simple product rule:
\[ \int (f(x) \cdot g(x)) \, dx \neq (\int f(x) \, dx) \cdot (\int g(x) \, dx), \]
Check that this is indeed not true by using \( f(x) = x \) and \( g(x) = x \), and computing both sides of the above equation.
For simplicity, we take as 0 all arbitrary constants that arise from indefinite integration.
\[\int (f(x) \cdot g(x)) \, dx = \int (x \cdot x) \, dx \]
\[= \int x^2 \, dx \]
\[= \frac{1}{3}x^3 \]
\[\neq x^4 \]
\[= (x^2) \cdot (x^2) \]
\[= (\int x \, dx) \cdot (\int x \, dx) \]
\[= (\int f(x) \, dx) \cdot (\int g(x) \, dx) \]