Problem Set 4 Solutions

1. (a) 
   \[ y(x) = \cos(x) \sin(x) \] 
   \[ y'(x) = \frac{1}{32} \sin(2x) \] 
   \[ y''(x) = 5\cos(x) \sin(x) \sin^2(x) \cos^2(x) = \frac{5}{16} \sin^4(2x) \cos(2x) \] 

(b) 
   \[ y'(x) = \frac{1}{\ln(5)} \left[ \frac{4x}{2x^2 - 6} + \frac{1}{x + 7} \right] \] 

(c) 
   \[ y'(x) = -8x \tan(4x^2) \] 

(d) 
   \[ y'(x) = e^{\tan(x)} \sec^2(x) \] 

(e) 
   \[ y'(x) = -\frac{1}{x^2} \cos\left(\frac{1}{x}\right) \] 

2. (a) 
   \[ y = x^3 + x^2 + 5x + 4 \] 
   \[ y' = 3x^2 + 2x + 5, \text{ which doesn't vanish for any real } x \] 
   \[ y'' = 6x + 2, \text{ which vanishes at } x = -\frac{1}{3} \] 
   The graph has no minima, since \( y' \) doesn't vanish, and it is always increasing. for \( x < -\frac{1}{3} \), the graph curves down (it’s concave), and for \( x > -\frac{1}{3} \) it curves up (it’s convex).

(b) 
   \[ y = e^{x^2} \] 
   \[ y' = 2xe^{x^2} \] 
   \[ y'' = (4x^2 + 2)e^{x^2} \] 
   The derivative vanishes just at \( x = 0 \). The value of \( y'' \) is greater than 0 everywhere so the graph curves up (it’s convex).
(c) 

\[ y = \frac{x - 3}{x^3 - 3x^2 - 9x + 27} = \frac{x - 3}{(x - 3)(x^2 - 9)} = \frac{1}{x^2 - 9} = \frac{1}{6} \left( \frac{1}{x - 3} - \frac{1}{x + 3} \right) \]

\[
y' = -\frac{2x}{(x^2 - 9)^2} = \frac{1}{6} \left( -\frac{1}{(x - 3)^2} + \frac{1}{(x + 3)^2} \right) \]

\[
y'' = \frac{1}{3} \left( \frac{1}{(x - 3)^3} - \frac{1}{(x + 3)^3} \right) \]

The derivative vanishes only at \( x = 0 \). The second derivative is negative at \( x = 0 \), so there is a local maximum there. The function is increasing for negative \( x \) and decreasing for positive \( x \). There are vertical asymptotes at \( x = \pm 3 \). The function tends to 0 as \( x \) goes to \( \pm \infty \).

3. Let \( x \) denote the number of dollars the price is reduced. The new sale price is \( 16 - x \) dollars and the profit per book is \( 10 - x \) dollars. The total number of books sold is estimated to be \( 180 + 30x \) so the total profit is

\[
\text{Profit} = (180 + 30x)(10 - x). \]

The derivative is

\[
\text{Profit}' = -(180 + 30x) + (10 - x)30 \]

and it vanishes at \( x = 2 \). The optimal price of the book is therefore

\[
\text{Best Price} = \$14 \]
(if x didn’t work out to be a whole number we would have to round up or down, depending on which gave the most profit)

4. If the Height of the box is \( x \) centimeters, then the base of the box will have dimensions \((10 - 2x)\) by \((20 - 2x)\). So the total volume is,

\[
Volume(x) = x(10 - 2x)(20 - 2x) = 4(x^3 - 15x^2 + 50x)
\]

Differentiating,

\[
Volume'(x) = 4(3x^2 - 30x + 50)
\]

Using the quadratic formula, we find this vanishes when

\[
x = 5 \pm \frac{5}{3}\sqrt{3}
\]

Since \( 10 - 2x \) has to be positive, \( x \) must be less than 5, so \( x \) must be \( 5 - \frac{5}{3}\sqrt{3} \) which is approximately 2.11. Plugging in to the volume formula, we get

\[
\text{Max. Volume} = 192.45 \text{ cm}^3
\]