Problem Set #3 Solutions

1. 
   a) \( y' = 12x^3 + 6x^2 - 2x + 4 \)
   b) \( y' = 36x^5 + 36x^3 - 12x^2 - 20x - 6 \)
   c) \( y' = \frac{-6x^2 + 28x + 12}{(2x^2 + 4)^2} \)
   d) \( y' = 50(10x-2)^4 \cdot (3x^2 - 1)^2 + 12(10x-2)^5 \cdot (3x^2 - 1) \cdot x \)

   OR when factored
   \[ 32 \cdot (3x^2 - 1) \cdot (135x^2 - 12x - 25) \cdot (5x - 1)^4 \]

   OR when expanded
   \[ 8100000x^8 - 1680000x^6 - 664000x^4 - 720000x^2 + 3168000x^5 - 209152x^3 + 105600x^2 - 15616x + 800 \]

   e) \( y' = \sec(\theta) \cdot \tan(\theta) \cdot \csc(\theta) - \sec(\theta) \cdot \csc(\theta) \cdot \cot(\theta) \)

   OR when simplified to \( \cos \)
   \[ y'' = \frac{2 \cdot \cos(\theta)^2 - 1}{\cos(\theta)^2 \cdot (\cos(\theta)^2 - 1)} \]

   f) \( y' = 1 + \tan^2 \theta \) \hspace{1cm} OR \hspace{1cm} \( y' = \sec^2 \theta \).

   g) \( y' = 5e^{5x+7} \)

   h) \[ y' = \frac{1}{3} \left( \frac{6 - x}{(4x+2)} - \frac{x^2}{(4x+2)^2} \cdot (4x+2) \right) \]

   which simplifies to
   \[ y' = \frac{2 \cdot (x+1)}{x \cdot (2x+1)} \]
h) In this problem, it is useful to square both sides of the equation before carrying out the implicit differentiation.

\[ y^6 = 2xy - 4xy^2 \]

Differentiating, and solving for \( \frac{dy}{dx} \), we get

\[ \frac{dy}{dx} = \frac{y(1-2y)}{3y^5 + 4xy - x} \]

j) By the chain rule, we get

\[ y' = 5 \cdot \left( \frac{3}{x^2 + 4} \right)^{\frac{3}{2}} \cdot x \]

2. Taking the derivative, we get

\[ f'(x) = 3ax^2 + 2bx + c \]

We now set this equation to zero and solve using the quadratic formula

\[ x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \]

We see that the number of horizontal tangents is dependent on the number of real solutions.

a) If \( 4b^2 - 12ac > 0 \)

Then the quantity under the square root is positive and there are two roots (or horizontal tangents)

b) If \( 4b^2 - 12ac = 0 \)

Then the quantity under the square root is zero and there is a single root \((-b/3a)\) and a single horizontal tangent.
c) If \(4b^2 - 12ac < 0\)

Then the quantity under the square root is negative, and we have a complex number which means there are no real roots (or horizontal tangents).

3. If \(f(x) = x + 2\sin(x)\)

Then \(f'(x) = 1 + 2\cos(x)\)

So, the derivative is simply a vertically shifted cosine function with a magnitude of 2. This being the case, we know that it will periodically cross the x-axis (y=0). Setting the derivative to zero, we get

\[
\cos(x) = -\frac{1}{2} \quad \text{and} \quad \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \frac{4\pi}{3}
\]

And since this is a periodic function repeating every \(2\pi\) radians, there are horizontal tangents at

\[
x = \begin{cases} 
\frac{2\pi}{3} + 2n\pi \\
\frac{4\pi}{3} + 2n\pi 
\end{cases}
\quad \text{where } n \text{ is an integer}
\]

4. If \(f(x) = \frac{x^3 + x}{x - 1}\)

Then \(f'(x) = \frac{2x^3 - 3x^2 - 1}{(x - 1)^2}\)

And so \(f'(2) = 3\)

Using the point slope form, the tangent line at the point (2,10) is

\(y - 10 = 3(x - 2)\) \quad \text{which in standard form is}

\(y = 3x + 4\)
5. If \( f(x) = \frac{6}{x+2} \)

Then \( f'(x) = -\frac{6}{(x+2)^2} \)

So, \( f'(1) = -\frac{2}{3} \)

Using the point slope form and simplifying, the tangent line at \((1,2)\) is

\[ y = \frac{-2}{3}x + \frac{8}{3} \]

We know that the slope of the normal to the tangent is the negative reciprocal of the slope of the tangent (\(3/2\) in this case) so using the point slope form and simplifying, the normal to the tangent is

\[ y = \frac{3}{2}x + \frac{1}{2} \]