

PROBLEM SET 2. PROBLEMS FROM LECTURE 2.

**Reading.** *Quick Calculus*, pp. 50–97.

**Supplementary reading.** Simmons, Chapter 2, sections 2.1–2.5. Read section 2.6 if you are interested in some applications of the derivative.

1. Compute the following limits.

(a)  $\lim_{\theta \rightarrow 0} \frac{\sin(5\theta)}{\theta}$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\sin(5\theta)}{\theta} &= 5 \left( \lim_{\theta \rightarrow 0} \frac{\sin(5\theta)}{5\theta} \right) \\ &= 5 \left( \lim_{\gamma \rightarrow 0} \frac{\sin(\gamma)}{\gamma} \right) \\ &= 5 \end{aligned}$$

Here, we used the fact that  $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1$ , and made the variable substitution  $\gamma = 5\theta$ .

(b)  $\lim_{\theta \rightarrow 0} \frac{\sin(3\theta)}{\sin(4\theta)}$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\sin(3\theta)}{\sin(4\theta)} &= \lim_{\theta \rightarrow 0} \frac{\frac{\sin(3\theta)}{\theta}}{\frac{\sin(4\theta)}{\theta}} \\ &= \frac{3}{4} \end{aligned}$$

(c)  $\lim_{x \rightarrow \infty} \frac{x}{x^2+1}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x}{x^2+1} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

(d)  $\lim_{x \rightarrow \infty} \frac{2x^2+3x}{3x^2-2}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^2+3x}{3x^2-2} &= \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x}}{3 - \frac{2}{x^2}} \\ &= \frac{2}{3} \end{aligned}$$

2. Where are the following functions discontinuous?

(a)  $\frac{x}{x^2+1}$

(b)  $\frac{1}{x^2+x-6}$

(c)  $\frac{x^3+x}{x^2+1}$

(d)  $\frac{x^2+2x}{x^3+2x^2-x-2}$

Functions (a) and (c) are continuous everywhere. Function (b) is discontinuous at  $x = 3$  and  $x = -2$ . Function (d) is discontinuous at  $x = 1$ ,  $x = -1$  and  $x = -2$ .

3. Use the definition of the derivative to show that for  $f(x) = ax^2 + bx + c$ , for constants  $a, b, c \in \mathbb{R}$ , the derivative is  $f'(x) = 2ax + b$ .

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(a(x + \Delta x)^2 + b(x + \Delta x) + c) - (ax^2 + bx + c)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(2ax\Delta x + (\Delta x)^2 + b\Delta x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} 2ax + \Delta x + b \\
 &= 2ax + b
 \end{aligned}$$

4. Use the definition of the derivative to find the derivative of the function  $f(x) = \frac{x}{x+1}$ .

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x}{x+\Delta x+1} - \frac{x}{x+1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(x+\Delta x)(x+1) - (x)(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{(x^2+x\Delta x+x+\Delta x) - (x^2+x\Delta x+x)}{(x^2+x\Delta x+2x+\Delta x+1)}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{(x^2+x\Delta x+2x+\Delta x+1)}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{(x^2 + x\Delta x + 2x + \Delta x + 1)} \\
 &= \frac{1}{x^2 + 2x + 1}
 \end{aligned}$$

Note that at  $x = -1$ , the function is undefined and the limit does not exist.

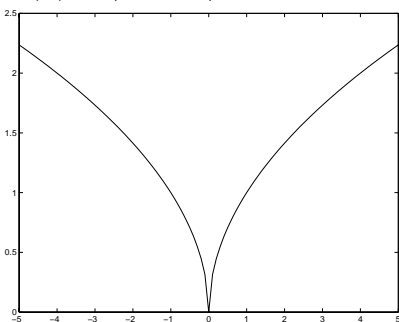
5. Use the definition of the derivative and the angle summation formula to compute the derivative of  $f(\theta) = \cos(\theta)$ .

$$\begin{aligned}
f'(\theta) &= \lim_{\Delta\theta \rightarrow 0} \frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta \rightarrow 0} \frac{(\cos(\theta)\cos(\Delta\theta) - \sin(\theta)\sin(\Delta\theta)) - \cos(\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta \rightarrow 0} \frac{\cos(\theta)(\cos(\Delta\theta) - 1) - \sin(\theta)\sin(\Delta\theta)}{\Delta\theta} \\
&= \lim_{\Delta\theta \rightarrow 0} \left[ \cos(\theta) \frac{\cos(\Delta\theta) - 1}{\Delta\theta} - \sin(\theta) \frac{\sin(\Delta\theta)}{\Delta\theta} \right] \\
&= \cos(\theta) \lim_{\Delta\theta \rightarrow 0} \frac{\cos(\Delta\theta) - 1}{\Delta\theta} - \sin(\theta) \lim_{\Delta\theta \rightarrow 0} \frac{\sin(\Delta\theta)}{\Delta\theta} \\
&= \cos(\theta)(0) - \sin(\theta)(1) \\
&= -\sin(\theta)
\end{aligned}$$

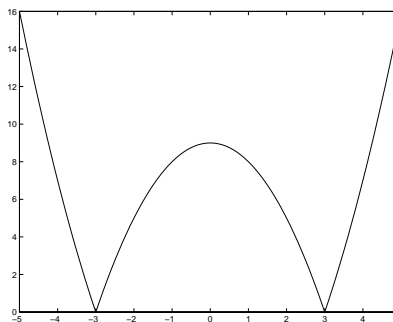
6. Sketch the graph of the following two functions. For each, state where it is not differentiable.

(a)  $f(x) = \sqrt{|x|}$ .

(b)  $f(x) = |x^2 - 9|$ .



$$f(x) = \sqrt{|x|}$$



$$f(x) = |x^2 - 9|$$

$f(x) = \sqrt{|x|}$  is discontinuous only at  $x = 0$ .  $f(x) = |x^2 - 9|$  is discontinuous at  $x = 3$  and  $x = -3$ .

7. Let  $f(x) = \begin{cases} x^2 & \text{if } x \leq -1, \\ mx + b & \text{if } x > -1. \end{cases}$  What must  $m$  and  $b$  be for  $f(x)$  to be differentiable at all points?

At  $x = -1$ , we need the two “branches” of the function to have identical derivatives and values. Noting that  $\frac{d}{dx}x^2 = 2x$  and  $\frac{d}{dx}(mx + b) = m$ , the first condition (equivalence of derivatives) requires that these be equal when  $x = -1$ , or, equivalently, that  $m = -2$ . Then, using  $m = -2$ , the second condition (equivalence of values) implies that  $b = 3$ .

8. A penny is dropped off a ledge on the World Trade Center in New York City. The ledge is 1024 feet above the ground. The penny falls a distance of  $s = 16t^2$  feet in  $t$  seconds.

(a) How long does the penny fall before it hits the ground?

$$16t^2 = 1024 \rightarrow t^2 = 64 \rightarrow t = 8$$

- (b) **What is the average velocity at which the penny falls during the first three seconds?**

The average velocity  $AV$  is given by

$$AV = \frac{s(3) - s(0)}{t(3) - t(0)} = \frac{16(3^2) - 0}{3 - 0} = \frac{144}{3} = 46$$

The average velocity during the first three seconds is 46 feet per second.

9. **With the same situation as in Problem 8, answer the following questions.**

- (a) **What is the average velocity at which the penny falls during the last four seconds?**

The average velocity  $AV$  is given by

$$AV = \frac{s(8) - s(4)}{t(8) - t(4)} = \frac{16(8^2) - 16(4^2)}{8 - 4} = \frac{768}{4} = 192$$

The average velocity during the last four seconds is 192 feet per second.

- (b) **What is the instantaneous velocity of the penny when it hits the ground?**

At any given time, the instantaneous velocity is given by  $v(t) = 32t$  (velocity is the derivative of distance with respect to time). Therefore, at  $t = 8$ , the instantaneous velocity is  $32(8) = 256$ .

10. **An oil tank is to be drained for cleaning. There are  $V$  gallons of oil left in the tank after  $t$  minutes of draining, where  $V = 50(40 - t)^2$ .**

- (a) **What is the average rate at which oil drains out of the tank during the first 20 minutes?**

The average rate of draining  $AR$  is given by

$$AR = \frac{V(20) - V(0)}{20 - 0} = \frac{50(40 - 20)^2 - 50(40 - 0)^2}{20} = -3000$$

The average rate of drainage during the first 20 minutes is 3,000 gallons per minute.

- (b) **What is the rate at which oil is flowing out of the tank 20 minutes after draining begins?**

The instantaneous change in the amount of oil in the tank at time  $t$  (for  $0 \leq t \leq 40$ ) is

$$\frac{dV}{dt} = -100(40 - t)$$

At  $t = 20$ , the rate of change is -2,000 gallons per minute; 2,000 gallons per minute are flowing out of the tank.